MTH 166

Lecture-32

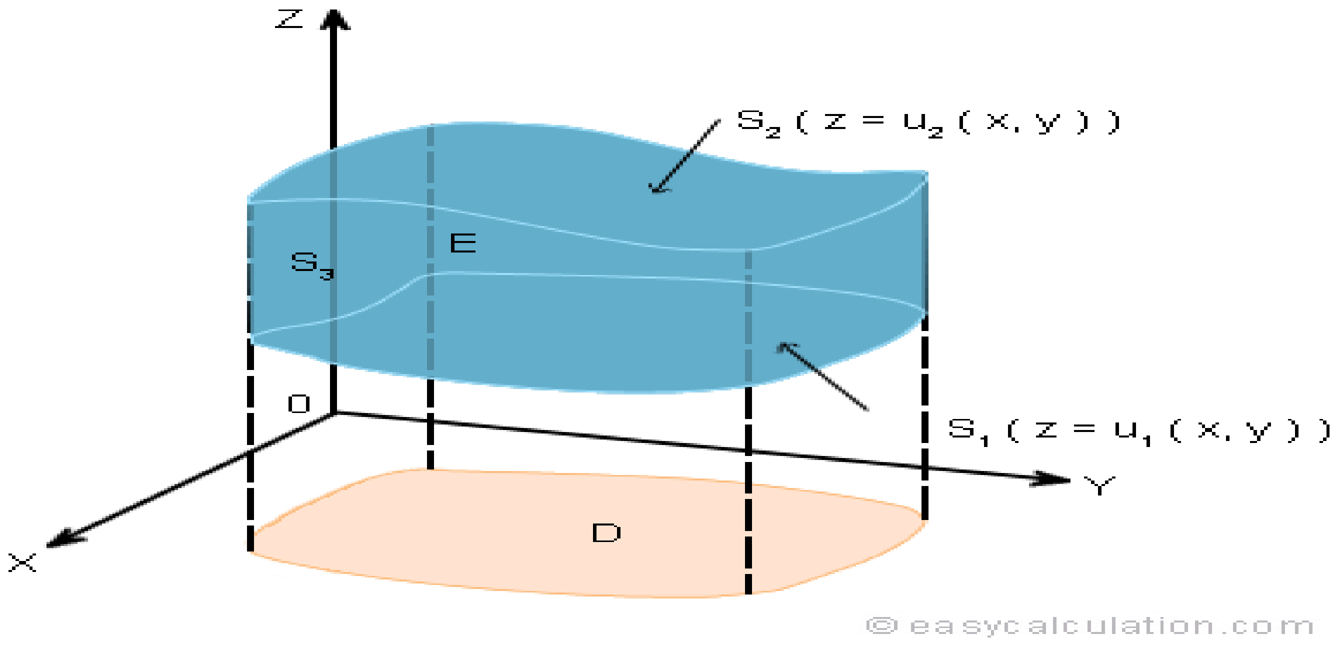
Surface Area and Surface Integral

Area Element and Surface Area

1. If the projection of the surface $z = f(x, y)$is taken on xy −plane.

Area element, $dA = \sqrt{1 + {f_x}^2 + {f_y}^2}$

Surface Area, $A = \iint_R \; dA = \iint_R \; \sqrt{1 + {f_x}^2 + {f_y}^2} dxdy$



2. If the projection of the surface x $= g(y, z)$is taken on yz $-$plane.

Area element, $dA = \sqrt{1 + {g_y}^2 + {g_z}^2}$

Surface Area, $A = \iint_R \; dA = \iint_R \; \sqrt{1 + {g_y}^2 + {g_z}^2}dxdy$

3. If the projection of the surface y $= h(z, x)$is taken on zx $-$plane.

Area element, $dA = \sqrt{1 + {h_z}^2 + {h_x}^2}$

Surface Area, $A = \iint_R \; dA = \iint_R \; \sqrt{1 + {h_z}^2 + {h_x}^2}dxdy$

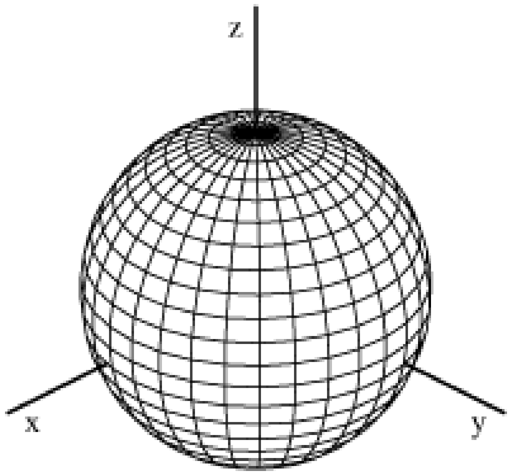
Important Questions from MCQ Point of View:

Problem 1. Find the surface area of $x^2 + y^2 + z^2 = a^2$

Solution: Since it is sphere of radius a

So, Surface area of sphere, $S = 4\pi r^2$

$\Rightarrow S = 4\pi(a)^2$ **Answer.**

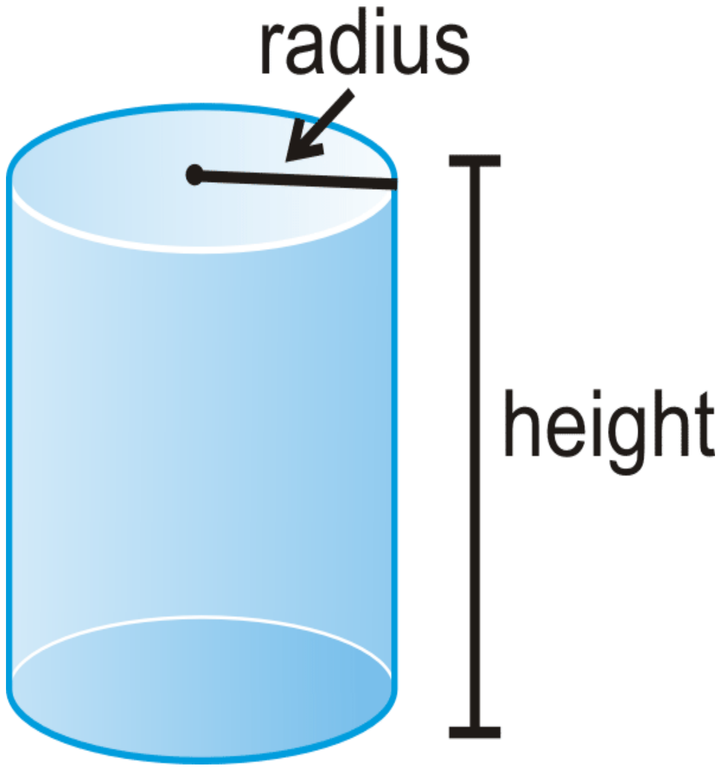


Problem 2. Find the surface area of $x^2 + y^2 = 16, 0 \leq z \leq 2$

Solution: Since it is cylinder of radius 4 and height 2.

So, Surface area of cylinder, $S = 2\pi rh$

$\Rightarrow S = 2\pi(4)(2) = 16\pi$ **Answer.**



Problem 3. Find the surface area of $z^2 = x^2 + y^2, 0 \leq z \leq 4$

Solution: Since it is a cone of height h = 4 and radius of topmost part r = 4.

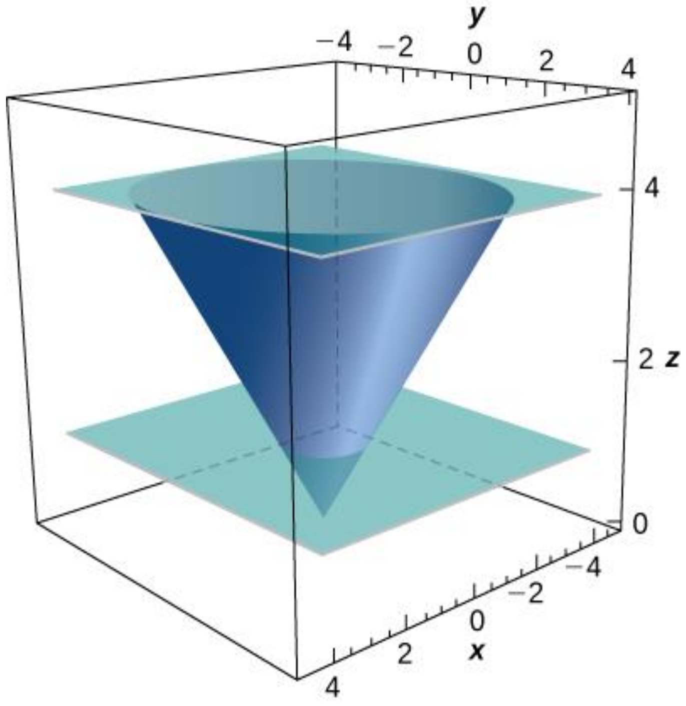
Here slanting height, $l = \sqrt{h^2 + r^2}$

$\Rightarrow l = \sqrt{(4)^2 + (4)^2} = 4\sqrt{2}$

So, Surface area of cone, $S = \pi r l$

$\Rightarrow S = \pi(4)\left(4\sqrt{2}\right)$

$\Rightarrow S = 16\sqrt{2}\pi$ **Answer.**



Surface Integral

The expression for surface integral is: $I = \iint_S \; f(x,y,z)dA$

Where dA is the area element and it can be determined from one of the following ways:

If projection is taken on xy-plane (z=0 plane), then: $dA = \frac{dxdy}{\hat{n}.\hat{k}}$

If projection is taken on yz-plane (x=0 plane), then: $dA = \frac{dydz}{\hat{n}.\hat{\imath}}$

If projection is taken on zx-plane (y=0 plane), then: $dA = \frac{dzdx}{\hat{n}.\hat{\jmath}}$

Problem 4. Evaluate the surface integral $\iint_S f(x,y,z)dA$ where $f = 6xyz$ and S is the portion of the plane $x + y + z = 1$ in the first octant.

Solution: Let g = x + y + z = 1

$$\Longrightarrow g_x = 1, g_y = 1, g_z = 1$$

Now, $\vec{\nabla} g = g_x \hat{\imath} + g_y \hat{\jmath} + g_z \hat{k} = \hat{\imath} + \hat{\jmath} + \hat{k}$

$$\hat{n} = \frac{\vec{\nabla} g}{|\vec{\nabla} g|} = \frac{(\hat{\imath}+\hat{\jmath}+\hat{k})}{\sqrt{1+1+1}}$$

$$\Rightarrow \hat{n} = \frac{(\hat{\imath}+\hat{\jmath}+\hat{k})}{\sqrt{3}}$$

Let projection of Surface S be taken on xy-plane:

$$z = 1 - x - y$$

$$dA = \frac{dxdy}{\hat{n}.\hat{k}} = \frac{dxdy}{\frac{(\hat{\imath}+\hat{\jmath}+\hat{k})}{\sqrt{3}}.\hat{k}}$$

$$\Rightarrow dA = \frac{dxdy}{\frac{1}{\sqrt{3}}}$$

For limits: On xy-plane(z = 0)

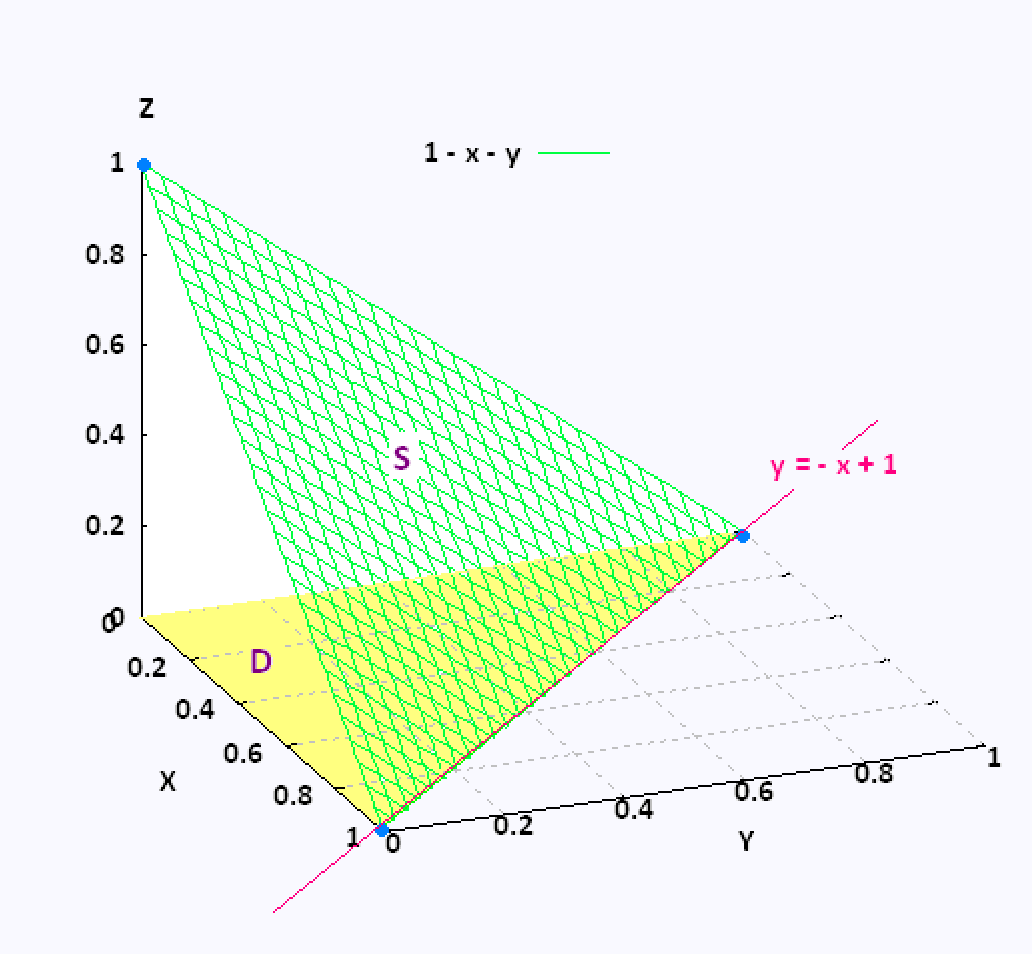
So, x + y + 0 = 1 $\Rightarrow y = (1 - x)$

$$0 \leq y \leq (1 - x)$$

On x-axis (y = 0, z = 0)

So, x + 0 + 0 = 1 $\Rightarrow x = 1$

$$0 \leq x \leq 1$$



So, required surface integral is: $I = \iint_S f(x,y,z)dA$

$$\Rightarrow I = \iint_S (6xyz)\frac{dxdy}{\frac{1}{\sqrt{3}}}$$

$$= 6\sqrt{3}\int_{x=0}^{1}\int_{y=0}^{1-x} xy(1-x-y)dxdy$$

$$= 6\sqrt{3}\int_{x=0}^{1}\int_{y=0}^{1-x}(xy - x^2y - xy^2)dydx$$

On simplification, we get:

$I = \frac{\sqrt{3}}{20}$ **Answer.**


THANK
you